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Monotone iterative algorithms for a nonlinear singularly perturbed parabolic problem

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Abstract

This paper deals with discrete monotone iterative algorithms for solving a nonlinear singularly perturbed parabolic reaction–diffusion problem. Firstly, the monotone method (known as the method of lower and upper solutions) is applied to computing a nonlinear difference scheme obtained after discretisation of the continuous problem. Secondly, a monotone domain decomposition algorithm based on a modification of the Schwarz alternating method is constructed. This monotone algorithm solves only linear discrete systems at each iterative step of the iterative process. The rate of convergence of the monotone domain decomposition algorithm is estimated. Numerical experiments are presented.

Keywords: Parabolic reaction–diffusion problem; Boundary layers; Monotone method; Monotone Schwarz alternating algorithm

1. Introduction

We are interested in monotone Schwarz alternating algorithms for solving the nonlinear reaction –diffusion problem

$$-\mu^2(u_{xx}+u_{yy})+u_t=-f(P,t,u),$$
$$P=(x,y),\quad (P,t)\in Q=\Omega\times(0,T],\quad \Omega=\{0<x<1,0<y<1\},$$
$$f_u(P,t,u)\geqslant 0,\quad (P,t,u)\in\bar{Q}\times(-\infty,\infty),\quad (f_u\equiv\partial f/\partial u), \tag{1}$$

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where μ is a small positive parameter. The initial-boundary conditions are defined by

$$u(P,t)=g(P,t), \qquad (P,t)\in\partial\Omega\times(0,T], \qquad u(P,0)=u^0(P), \quad P\in\bar{\Omega},$$

where $\partial\Omega$ is the boundary of Ω. The functions $f(P,t,u)$, $g(P,t)$ and $u^0(P)$ are sufficiently smooth. Under suitable continuity and compatibility conditions on the data, a unique solution $u(P,t)$ of (1) exists (see [8] for details). For $\mu\ll 1$, problem (1) is singularly perturbed and characterized by the boundary layers of width $O(\mu|\ln\mu|)$ at the boundary $\partial\Omega$ (see [2] for details).

In the study of numerical solutions of nonlinear singularly perturbed problems by the finite difference method, the corresponding discrete problem is usually formulated as a system of nonlinear algebraic equations. A major point about this system is to obtain reliable and efficient computational algorithms for computing the solution. In the case of the parabolic problem (1), the implicit method is usually in use. On each time level, this method leads to a nonlinear system (with M-matrix and diagonal operator defined by f) which requires some kind of iterative scheme for the computation of numerical solutions. A fruitful method for the treatment of these nonlinear systems is the method of upper and lower solutions and its associated monotone iterations (in the case of "unperturbed" problems with reaction–diffusion equations see [15,16] and references therein). Since the initial iteration in the monotone iterative method is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution (see [5] for details), this method eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.

Iterative domain decomposition algorithms based on Schwarz-type alternating procedures have received much attention for their potential as efficient algorithms for parallel computing. Lions [11] proved convergence of a multiplicative Schwarz method for Poisson's equation using the monotone method. In [12], some Schwarz methods for nonlinear elliptic problems using the monotone method were considered. Both Lions [11] and Lui [12] examined the theoretical convergence properties of continuous, but not discrete, Schwarz methods, and the two important points in studying monotone Schwarz methods concerning construction of initial lower or upper solutions (initial guesses) and estimates of rates of convergence were omitted. In [5], for solving nonlinear reaction–diffusion problems of elliptic type, we proposed the discrete iterative algorithm which combines the monotone approach and the iterative domain decomposition method based on the Schwarz alternating procedure.

We mention here that in the context of solving systems of nonlinear equations, the monotone iterative method belongs to the class of methods based on convergence under partial ordering (see Chapter 13 in [14] for details). In recent years, the monotone iterative method has received a great deal of attention for solving more general differential problems approximated by the following finite-dimensional problem:

$$AU+\sum_{k=1}^{s}B_k\Upsilon_k=f, \qquad \Upsilon_k\in C_kU$$

with M-matrices $A, B_k, k=1,\dots,s$, and diagonal maximal monotone (multivalued) operators $C_k, k=1,\dots,s$. Mesh approximations of free and moving boundary problems with several sets of constraints, nonlinear relations and unknown boundaries lead to the above equation. Iterative methods including Schwarz alternating methods for the problem $AU+\Upsilon=f$, $\Upsilon\in CU$ have been investigated in [1,7,11] (see also references therein). The geometric convergent rate of the iterative methods for the problem with several M-matrices has been studied in [10]. The iterative methods for problems with nonlinear M-mappings A and B have been investigated in [9].

In this paper, we consider a monotone domain decomposition algorithm based on the multidomain modification of the discrete Schwarz alternating method proposed in [4] and on the monotone approach from [5]. Here the computational domain in the space variables is partitioned into many nonoverlapping subdomains with interface Γ. Small interfacial subdomains are introduced near the interface Γ, and approximate boundary values computed on Γ are used for solving problems on nonoverlapping subdomains. Thus, this approach may be considered as a variant of a block Gauss–Seidel iteration (or in the parallel context as a multicoloured algorithm) for the subdomains with a Dirichlet–Dirichlet coupling through the interface variables.

The structure of the paper is as follows. In Section 2, we consider a monotone iterative method for solving the implicit difference scheme which approximates the nonlinear problem (1). In Section 3, we construct a monotone domain decomposition algorithm and investigate the rate of convergence of this algorithm. The final Section 4 presents results of numerical experiments for the proposed algorithm.

2. Monotone iterative method

On $\bar{Q}$ introduce a rectangular mesh $\bar{\Omega}^h \times \bar{\Omega}^\tau$, $\bar{\Omega}^h = \bar{\Omega}^{hx} \times \bar{\Omega}^{hy}$:

$$\bar{\Omega}^{hx} = \{x_i,\ 0 \leqslant i \leqslant N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\},$$

$$\bar{\Omega}^{hy} = \{y_j,\ 0 \leqslant j \leqslant N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\},$$

$$\bar{\Omega}^{\tau} = \{t_k = k\tau,\ 0 \leqslant k \leqslant N_\tau,\ N_\tau \tau = T\}.$$

For a mesh function $U(P,t)$, we use the implicit difference scheme

$$\mathscr{L}^h U(P,t) + \frac{1}{\tau}[U(P,t) - U(P,t-\tau)] = -f(P,t,U), \quad (P,t) \in \Omega^h \times \Omega^\tau,$$

$$U(P,t) = g(P,t), \qquad (P,t) \in \partial\Omega^h \times \Omega^\tau, \qquad U(P,0) = u^0(P), \quad P \in \bar{\Omega}^h, \tag{2}$$

where $\mathscr{L}^h U(P,t)$ is defined by

$$\mathscr{L}^h U = -\mu^2 (D_+^x D_-^x + D_+^y D_-^y) U.$$

$D_+^x D_-^x U(P,t)$, $D_+^y D_-^y U(P,t)$ are the central difference approximations to the second derivatives

$$D_+^x D_-^x U_{ij}^k = (\hbar_{xi})^{-1}[(U_{i+1,j}^k - U_{ij}^k)(h_{xi})^{-1} - (U_{ij}^k - U_{i-1,j}^k)(h_{xi-1})^{-1}],$$

$$D_+^y D_-^y U_{ij}^k = (\hbar_{yj})^{-1}[(U_{i,j+1}^k - U_{ij}^k)(h_{yj})^{-1} - (U_{ij}^k - U_{i,j-1}^k)(h_{yj-1})^{-1}],$$

$$\hbar_{xi} = 2^{-1}(h_{xi-1} + h_{xi}), \quad \hbar_{yj} = 2^{-1}(h_{yj-1} + h_{yj}),$$

where $U_{ij}^k \equiv U(x_i, y_j, t_k)$.

Now, we construct an iterative method for solving the nonlinear difference scheme (2) which possesses the monotone convergence. This method is based on the approach from [2]. Represent the difference equation from (2) in the equivalent form

$$\mathscr{L}U(P,t) = -f(P,t,U) + \frac{U(P,t-\tau)}{\tau}, \quad \mathscr{L}U(P,t) \equiv \mathscr{L}^h U(P,t) + \frac{U(P,t)}{\tau},$$

and for τ fixed, on $\bar{\Omega}^h$ introduce the linear difference problem

$$\begin{aligned} &\mathscr{L}W(P) + c(P)W(P) = F(P), \quad P \in \Omega^h, \\ &W(P) = W^0(P), \quad P \in \partial\Omega^h, \qquad c(P) \geqslant c_0 > 0, \quad P \in \bar{\Omega}^h, \end{aligned} \tag{3}$$

where c_0 is a constant. Now, we formulate a discrete maximum principle for the difference operator $\mathscr{L} + c$ and give an estimate of the solution to (3).

Lemma 1. (i) *If $W(P)$ satisfies the conditions*

$$\mathscr{L}W(P) + c(P)W(P) \geqslant 0(\leqslant 0), \quad P \in \Omega^h, \;\; W(P) \geqslant 0(\leqslant 0), \;\; P \in \partial\Omega^h,$$

then $W(P) \geqslant 0(\leqslant 0)$, $P \in \bar{\Omega}^h$.

(ii) *The following estimate of the solution to (3) holds true*

$$\|W\|_{\bar{\Omega}^h} \leqslant \max\left[\|W^0\|_{\partial\Omega^h}, \frac{\|F\|_{\Omega^h}}{c_0 + \tau^{-1}}\right], \tag{4}$$

where

$$\|W^0\|_{\partial\Omega^h} \equiv \max_{P \in \partial\Omega^h} |W^0(P)|, \quad \|F\|_{\Omega^h} \equiv \max_{P \in \Omega^h} |F(P)|.$$

The proof of the lemma can be found in [18].

Additionally, we assume that $f(P,t,u)$ from (1) satisfies the two-sided constraints

$$0 \leqslant f_u \leqslant c^*, \quad c^* = \text{const}. \tag{5}$$

We say that on a time level $t \in \Omega^\tau$, $\bar{V}(P,t)$ is an upper solution with a given function $V(P,t-\tau)$, if it satisfies

$$\begin{aligned} &\mathscr{L}\bar{V}(P,t) + f(P,t,\bar{V}) - \frac{V(P,t-\tau)}{\tau} \geqslant 0, \quad P \in \Omega^h, \\ &\bar{V}(P,t) = g(P,t), \quad P \in \partial\Omega^h. \end{aligned}$$

Similarly, $\underline{V}(P,t)$ is called a lower solution on a time level $t \in \Omega^\tau$ with a given function $V(P,t-\tau)$, if it satisfies the reversed inequality and the boundary condition.

The iterative solution $V(P,t)$ to (2) is constructed in the following way. On each time level $t \in \Omega^\tau$, we calculate n_* iterates $V^{(n)}(P,t)$, $P \in \bar{\Omega}^h$, $n = 1,\dots,n_*$ using the recurrence formulas

$$\begin{aligned} \mathscr{L}Z^{(n+1)}(P,t) + c^* Z^{(n+1)}(P,t) = &-[\mathscr{L}V^{(n)}(P,t) + f(P,t,V^{(n)}) \\ &-\tau^{-1}V(P,t-\tau)], \quad P \in \Omega^h, \end{aligned}$$

$$Z^{(n+1)}(P,t)=0, \quad P\in\partial\Omega^h, \; n=0,\dots,n_*-1,$$

$$V^{(n+1)}(P,t)=V^{(n)}(P,t)+Z^{(n+1)}(P,t), \quad P\in\bar{\Omega}^h,$$

$$V(P,t)\equiv V^{(n_*)}(P,t), \quad P\in\bar{\Omega}^h, \qquad V(P,0)=u^0(P), \quad P\in\bar{\Omega}^h, \tag{6}$$

where an initial guess $V^{(0)}(P,t)$ satisfies the boundary condition

$$V^{(0)}(P,t)=g(P,t), \quad P\in\partial\Omega^h.$$

Theorem 1. *Let $V(P,t-\tau)$ be given and $\bar{V}^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t)$ be upper and lower solutions corresponding to $V(P,t-\tau)$. Suppose that $f(P,t,u)$ satisfies (5). Then the upper sequence $\{\bar{V}^{(n)}(P,t)\}$ generated by (6) converges monotonically from above to the unique solution $\mathscr{V}(P,t)$ of the problem*

$$\mathscr{L}V(P,t)+f(P,t,V)-\frac{V(P,t-\tau)}{\tau}=0, \quad P\in\Omega^h,$$

$$V(P,t)=g(P,t), \quad P\in\partial\Omega^h, \tag{7}$$

the lower sequence $\{\underline{V}^{(n)}(P,t)\}$ generated by (6) converges monotonically from below to $V(P,t)$:

$$\underline{V}^{(0)}(P,t)\leqslant \underline{V}^{(n)}(P,t)\leqslant \underline{V}^{(n+1)}(P,t)\leqslant \mathscr{V}(P,t), \quad P\in\bar{\Omega}^h,$$

$$\mathscr{V}(P,t)\leqslant \bar{V}^{(n+1)}(P,t)\leqslant \bar{V}^{(n)}(P,t)\leqslant \bar{V}^{(0)}(P,t), \quad P\in\bar{\Omega}^h,$$

and the sequences converge with the linear rate $\rho=c^*/(c^*+\tau^{-1})$.

Proof. We consider only the case of the upper sequence. If $\bar{V}^{(0)}(P,t)$ is an upper solution, then from (6) we conclude that

$$\mathscr{L}Z^{(1)}(P,t)+c^*Z^{(1)}(P,t)\leqslant 0, \quad P\in\Omega^h, \qquad Z^{(1)}(P,t)=0, \quad P\in\partial\Omega^h.$$

From Lemma 1, by the maximum principle for the difference operator $\mathscr{L}+c^*$, it follows that $Z^{(1)}(P,t)\leqslant 0$, $P\in\bar{\Omega}^h$. Using the mean-value theorem and the equation for $Z^{(1)}$, we have

$$\mathscr{L}\bar{V}^{(1)}(P,t)+f(P,t,\bar{V}^{(1)})-\frac{V(P,t-\tau)}{\tau}=-[c^*-f_u^{(1)}(P,t)]\,Z^{(1)}(P,t), \tag{8}$$

where $f_u^{(1)}(P,t)\equiv f_u[P,t,\bar{V}^{(0)}(P,t)+\theta^{(1)}(P,t)Z^{(1)}(P,t)]$, $0<\theta^{(1)}(P,t)<1$. Since the mesh function $Z^{(1)}(P,t)$ is nonpositive on Ω^h and taking into account (5), we conclude that $\bar{V}^{(1)}(P,t)$ is an upper solution. By induction we obtain that $Z^{(n)}(P,t)\leqslant 0$, $P\in\bar{\Omega}^h$, $n=1,2,\dots$, and prove that $\{\bar{V}^{(n)}(P,t)\}$ is a monotonically decreasing sequence of upper solutions.

Now we shall prove that the monotone sequence $\{\bar{V}^{(n)}(P,t)\}$ converges to the solution of (7). Similar to (8), we obtain

$$\mathscr{L}\bar{V}^{(n)}(P,t)+f(P,t,\bar{V}^{(n)})-\frac{V(P,t-\tau)}{\tau}=-[c^*-f_u^{(n)}(P,t)]\,Z^{(n)}(P,t) \tag{9}$$

and from (6), it follows that $Z^{(n+1)}(P,t)$ satisfies the difference equation

$$\mathscr{L}Z^{(n+1)}(P,t)+c^*Z^{(n+1)}(P,t)=(c^*-f_u^{(n)})Z^{(n)}(P,t), \quad P\in\Omega^h.$$

Using (4) and (5), we conclude

$$\|Z^{(n+1)}(t)\|_{\bar{\Omega}^h} \leqslant \rho^n \|Z^{(1)}(t)\|_{\bar{\Omega}^h}, \quad \rho = \frac{c^*}{c^* + \tau^{-1}}. \tag{10}$$

This proves convergence of the upper sequence to the solution $\mathscr{V}$ of (7) with the linear rate ρ. In view of $\lim \bar{V}^{(n)} = \mathscr{V}$ as $n \to \infty$, we conclude that $\mathscr{V} \leqslant \bar{V}^{(n+1)} \leqslant \bar{V}^{(n)}$.

The uniqueness of the solution to (7) follows from estimate (4). Indeed, if by contradiction, we assume that there exist two solutions $\mathscr{V}_1$ and $\mathscr{V}_2$ to (7), then by the mean-value theorem, the difference $\delta\mathscr{V} = \mathscr{V}_1 - \mathscr{V}_2$ satisfies the following difference problem:

$$\mathscr{L}\delta\mathscr{V}(P,t) + f_u \delta\mathscr{V}(P,t) = 0, \quad P \in \Omega^h, \qquad \delta\mathscr{V}(P,t) = 0, \quad P \in \partial\Omega^h.$$

By (4), this leads to the uniqueness of the solution to (7). $\square$

Theorem 2. *Let $V^{(0)}(P,t)$ be an upper or lower solution in the iterative method (6), and let $f(P,t,u)$ satisfy (5). Suppose that on each time level the number of iterates n_* satisfies $n_* \geqslant 2$. Then the following estimate on convergence rate holds:*

$$\max_{t \in \Omega^\tau} \|V(t) - U(t)\| \leqslant C(\rho)^{n_*-1},$$

where $U(P,t)$ is the solution to (2) and constant C is independent of τ. Furthermore, on each time level the sequence $\{V^{(n)}(P,t)\}$ converges monotonically.

Proof. Introduce the notation

$$W(P,t) = U(P,t) - V(P,t),$$

where $V(P,t) \equiv V^{(n_*)}(P,t)$. Using the mean-value theorem, from (2), (9), conclude that $W(P,\tau)$ satisfies

$$\mathscr{L}W(P,\tau) + f_u(P,\tau)W(P,\tau) = [c^* - f_u^{(n_*)}(P,\tau)]Z^{(n_*)}(P,\tau), \quad P \in \Omega^h,$$

$$W(P,\tau) = 0, \quad P \in \partial\Omega^h,$$

where $f_u(P,\tau) \equiv f_u[P,\tau,U(P,\tau) + \theta(P,\tau)W(P,\tau)]$, $0 < \theta(P,\tau) < 1$, and we have taken into account that $V(P,0) = U(P,0)$. By (4), (5) and (10),

$$\|W(\tau)\| \leqslant c^* \tau \rho^{n_*-1} \|Z^{(1)}(\tau)\|.$$

Estimate $Z^{(1)}(P,\tau)$ from (6) by (4),

$$\|Z^{(1)}(\tau)\| \leqslant \tau \|\mathscr{L}V^{(0)}(\tau) + f(V^{(0)}) - \tau^{-1}u^0\| \leqslant C_1,$$

where C_1 is independent of τ. Thus,

$$\|W(\tau)\| \leqslant \tilde{C}_1 \tau \rho^{n_*-1}, \quad \tilde{C}_1 = c^* C_1, \tag{11}$$

where $\tilde{C}_1$ is independent of τ. Similarly, from (2), (9), it follows that

$$\mathscr{L}W(P,2\tau) + f_u(P,2\tau)W(P,2\tau) = \frac{W(P,\tau)}{\tau} + [c^* - f_u^{(n_*)}(P,2\tau)]Z^{(n_*)}(P,2\tau).$$

By (4),

$$\|W(2\tau)\| \leqslant \|W(\tau)\| + c^* \tau \rho^{n_*-1} \|Z^{(1)}(2\tau)\|.$$

Estimate $Z^{(1)}(P,2\tau)$ from (6) by (4),

$$\|Z^{(1)}(2\tau)\| \leqslant \tau\|\mathscr{L}V^{(0)}(2\tau) + f(V^{(0)}) - \tau^{-1}U(\tau)\| \leqslant C_2,$$

where C_2 is independent of τ. From here and (11), we conclude

$$\|W(2\tau)\| \leqslant (\tilde{C}_1 + \tilde{C}_2)\tau\rho^{n_*-1}, \quad \tilde{C}_2 = c^*C_2.$$

By induction, we prove

$$\|W(t_k)\| \leqslant \left(\sum_{l=1}^{k} \tilde{C}_l\right)\tau\rho^{n_*-1}, \quad k = 1,\ldots,N_\tau, \tag{12}$$

where all constants $\tilde{C}_l$ are independent of τ. Denoting

$$C_0 = \max_{1\leqslant l\leqslant N_\tau} \tilde{C}_l$$

and taking into account that $N_\tau\tau = T$, we prove the estimate in the theorem with $C = TC_0$. $\square$

Remark 1. Consider the following approach for constructing initial upper and lower solutions $\bar{V}^{(0)}(P,t)$ and $\underline{V}^{(0)}(P,t)$. Suppose that for t fixed, a mesh function $R(P,t)$ is defined on $\bar{\Omega}^h$ and satisfies the boundary condition $R(P,t) = g(P,t)$ on $\partial\Omega^h$. Introduce the following difference problems:

$$\mathscr{L}Z_q^{(0)}(P,t) = q|\mathscr{L}R(P,t) + f(P,t,R) - \tau^{-1}V(P,t-\tau)|, \quad P \in \Omega^h,$$

$$Z_q^{(0)}(P,t) = 0, \quad P \in \partial\Omega^h, \ q = 1,-1.$$

Then the functions $\bar{V}^{(0)}(P,t) = R(P,t) + Z_1^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t) = R(P,t) + Z_{-1}^{(0)}(P,t)$ are upper and lower solutions, respectively.

We check only that $\bar{V}^{(0)}(P,t)$ is an upper solution. From the maximum principle, it follows that $Z_1^{(0)}(P,t) \geqslant 0$ on $\bar{\Omega}^h$. Now using the difference equation for $Z_1^{(0)}$, we have

$$\mathscr{L}(R + Z_1^{(0)}) + f(R + Z_1^{(0)}) - \tau^{-1}V(P,t-\tau) = F(P,t) + |F(P,t)| + f_u^{(0)}Z_1^{(0)},$$

$$F(P,t) \equiv |\mathscr{L}R(P,t) + f(P,t,R) - \tau^{-1}V(P,t-\tau)|.$$

Since $f_u^{(0)} \geqslant 0$ and $Z_1^{(0)}$ is nonnegative, we conclude that $\bar{V}^{(0)}(P,t)$ is an upper solution.

Remark 2. Since the initial iteration in the monotone iterative method (6) is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the solution as we have suggested in the previous remark, this algorithm eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.

Remark 3. The implicit two-level difference scheme (2) is of the first order with respect to τ. From here and since $\rho \leqslant c^*\tau$, one may choose $n_* = 2$ to keep the global error of the monotone iterative method (6) consistent with the global error of the difference scheme (2).

3. Monotone domain decomposition algorithm

As for the monotone iterative method (6), we assume that $f(P,t,u)$ from (1) satisfies (5).

3.1. Statement and convergence of monotone algorithm

Consider decomposition of the domain $\bar{\Omega}$ into M nonoverlapping subdomains (vertical strips) $\bar{\Omega}_m$, $m=1,\dots,M$:

$$\Omega_m = \Omega_m^x \times (0,1), \qquad \Omega_m^x = (x_{m-1}, x_m), \quad x_0 = 0, \ x_M = 1,$$

$$\Gamma_m^b = \{x = x_{m-1}, 0 \leqslant y \leqslant 1\}, \qquad \Gamma_m^e = \{x = x_m, 0 \leqslant y \leqslant 1\},$$

$$\bar{\Omega}_{m-1} \cap \bar{\Omega}_m = \Gamma_m^b, \qquad \Gamma_m^b = \Gamma_{m-1}^e, \quad m = 2,\dots,M.$$

Thus, we can write down the boundary of Ω_m as

$$\partial\Omega_m = \Gamma_m^b \cup \Gamma_m^e \cup \Gamma_m^0, \qquad \Gamma_m^0 = \partial\Omega \cap \partial\Omega_m.$$

Additionally, introduce $(M-1)$ interfacial subdomains ω_m, $m=1,\dots,M-1$:

$$\omega_m = \omega_m^x \times (0,1), \qquad \omega_m^x = (x_m^b, x_m^e),$$

$$\omega_{m-1} \cap \omega_m = \emptyset, \qquad x_m^b < x_m < x_m^e, \quad m = 1,\dots,M-1.$$

The boundaries of ω_m are denoted by

$$\gamma_m^b = \{x = x_m^b, 0 \leqslant y \leqslant 1\}, \qquad \gamma_m^e = \{x = x_m^e, 0 \leqslant y \leqslant 1\}, \qquad \gamma_m^0 = \partial\Omega \cap \partial\omega_m.$$

Fig. 1 illustrates the x-section of the multidomain decomposition.

We now introduce meshes on $\bar{\Omega}_m$, $m=1,\dots,M$ and on $\bar{\omega}_m$, $m=1,\dots,M-1$. Suppose that the following set of mesh points belongs to mesh $\bar{\Omega}^h$:

$$\{x_m^b, x_m, x_m^e\}_{m=1}^{M-1} \subset \Omega^{hx},$$

then

$$\bar{\Omega}_m^h = \bar{\Omega}_m \cap \bar{\Omega}^h, \qquad \bar{\omega}_m^h = \bar{\omega}_m \cap \bar{\Omega}^h,$$

$$\Gamma_m^{hb,e,0} = \Gamma_m^{b,e,0} \cap \bar{\Omega}_m^h, \qquad \gamma_m^{hb,e,0} = \gamma_m^{b,e,0} \cap \bar{\omega}_m^h.$$

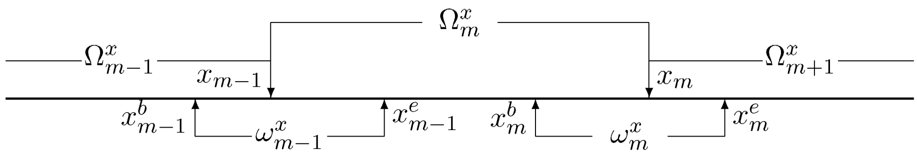

Fig. 1. The x-section of the multidomain decomposition.

Consider a parallel domain decomposition algorithm for solving problem (2). On each time level $t \in \Omega^\tau$, we calculate n_* iterates $V^{(n)}(P,t)$, $P \in \bar{\Omega}^h$, $n=1,\dots,n_*$. To find $V^{(n)}$, firstly, we solve problems on the nonoverlapping subdomains $\bar{\Omega}^h_m$, $m = 1,\dots,M$ with Dirichlet boundary conditions passed from the previous iterate. Then Dirichlet data are passed from these subdomains to the interfacial subdomains $\bar{\omega}^h_m$, $m=1,\dots,M-1$, and problems on the interfacial subdomains are computed. Finally, we piece together the solutions on the subdomains.

Step 0. Initialisation: On the mesh $\bar{\Omega}^h$, choose an upper or lower solution $V^{(0)}(P,t)$, $P \in \bar{\Omega}^h$ satisfying the boundary condition $V^{(0)}(P,t) = g(P,t)$ on $\partial\Omega^h$.

For $n = 1$ to n_* do Steps 1–3

Step 1. For $m=1$ to M do: On the subdomain $\bar{\Omega}^h_m$, compute the mesh function $Z^{(n)}_m(P,t)$, satisfying the difference scheme

$$\begin{aligned}\mathscr{L}Z^{(n)}_m(P,t) + c^* Z^{(n)}_m(P,t) = -[&\mathscr{L}V^{(n-1)}(P,t) + f(P,t,V^{(n-1)}) \\ &- \tau^{-1}V(P,t-\tau)], \quad P \in \Omega^h_m,\end{aligned}$$

$$Z^{(n)}_m(P,t) = 0, \qquad P \in \partial\Omega^h_m \tag{13}$$

and denote

$$V^{(n)}_m(P,t) = V^{(n-1)}(P,t) + Z^{(n)}_m(P,t), \qquad P \in \bar{\Omega}^h_m.$$

Step 2. For $m=1$ to $M-1$ do: On the interfacial subdomain $\bar{\omega}^h_m$, compute the difference problem

$$\begin{aligned}\mathscr{L}\tilde{Z}^{(n)}_m(P,t) + c^* \tilde{Z}^{(n)}_m(P,t) = -[&\mathscr{L}V^{(n-1)}(P,t) + f(P,t,V^{(n-1)}) \\ &- \tau^{-1}V(P,t-\tau)], \quad P \in \omega^h_m,\end{aligned}$$

$$\tilde{Z}^{(n)}_m(P,t) = \begin{cases} 0, & P \in \gamma^{h0}_m; \\ Z^{(n)}_m(P,t), & P \in \gamma^{hb}_m; \\ Z^{(n)}_{m+1}(P,t), & P \in \gamma^{he}_m \end{cases} \tag{14}$$

and denote

$$\tilde{V}^{(n)}_m(P,t) = V^{(n-1)}(P,t) + \tilde{Z}^{(n)}_m(P,t), \quad P \in \bar{\omega}^h_m.$$

Step 3. Compute the solution $V^{(n)}(P,t)$, $P \in \bar{\Omega}^h$ by piecing the solutions on the subdomains

$$V^{(n)}(P,t) = \begin{cases} V^{(n)}_m(P,t), & P \in \overline{\Omega^h_m \setminus (\omega^h_{m-1} \cup \omega^h_m)},\ m = 1,\dots,M; \\ \tilde{V}^{(n)}_m(P,t), & P \in \bar{\omega}^h_m,\ m = 1,\dots,M-1. \end{cases} \tag{15}$$

Step 4. Set up

$$V(P,t) = V^{(n_*)}(P,t), \quad P \in \bar{\Omega}^h. \tag{16}$$

Remark 4. We note that the original Schwarz alternating algorithm with overlapping subdomains is a purely sequential algorithm. To obtain parallelism, one needs a subdomain colouring strategy, so that a set of independent subproblems can be introduced. The proposed modification of the Schwarz algorithm is very suitable for parallel computing. Algorithm (13)–(16) can be carried out by parallel processing, since the M problems (13) for $V_m^{(n)}(P,t)$, $m=1,\dots,M$ and the $(M-1)$ problems (14) for $\tilde{V}_m^{(n)}(P,t)$, $m=1,\dots,M-1$ can be implemented concurrently.

Remark 5. Since the initial iteration in Algorithm (13)–(16) is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the solution, this algorithm eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.

Theorem 3. *Let $V(P,t-\tau)$ be given and $\bar{V}^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t)$ be upper and lower solutions corresponding to $V(P,t-\tau)$. Suppose that $f(P,t,u)$ satisfies (5). Then the upper sequence $\{\bar{V}^{(n)}(P,t)\}$ generated by (13)–(15) converges monotonically from above to the unique solution $\mathscr{V}(P,t)$ of problem (7), and the lower sequence $\{\underline{V}^{(n)}(P,t)\}$ generated by (13)–(15) converges monotonically from below to $\mathscr{V}(P,t)$:*

$$\underline{V}^{(0)}(P,t) \leqslant \underline{V}^{(n)}(P,t) \leqslant \underline{V}^{(n+1)}(P,t) \leqslant \mathscr{V}(P,t), \quad P \in \bar{\Omega}^h,$$

$$\mathscr{V}(P,t) \leqslant \bar{V}^{(n+1)}(P,t) \leqslant \bar{V}^{(n)}(P,t) \leqslant \bar{V}^{(0)}(P,t), \quad P \in \bar{\Omega}^h.$$

Proof. Consider the case of the upper sequence and suppose that $\bar{V}^{(n-1)}$ is an upper solution. By the maximum principle in Lemma 1, from (13) we have

$$Z_m^{(n)}(P,t) \leqslant 0, \quad P \in \bar{\Omega}_m^h. \tag{17}$$

Using the mean-value theorem, from (13) we obtain the difference problem for $V_m^{(n)}$ in the form

$$\mathscr{L}V_m^{(n)}(P,t) + f(P,t,V_m^{(n)}) - \frac{V(P,t-\tau)}{\tau} = -[c^* - f_u^{(n)}(P,t)]\,Z_m^{(n)}(P,t) \geqslant 0,$$
$$V_m^{(n)}(P,t) = \bar{V}^{(n-1)}(P,t), \quad P \in \partial\Omega_m^h, \tag{18}$$

where nonnegativeness of the right-hand side of the difference equation follows from (5) and (17). Taking into account that $\bar{V}^{(n-1)}$ is an upper solution, by the maximum principle in Lemma 1, from (14) and (17), it follows that

$$\tilde{Z}_m^{(n)}(P,t) \leqslant 0, \quad P \in \bar{\omega}_m^h. \tag{19}$$

Similar to (18), on the interfacial subdomain $\bar{\omega}_m^h$ we obtain the difference problem for $\tilde{V}_m^{(n)}$ in the form

$$\mathscr{L}\tilde{V}_m^{(n)}(P,t) + f(P,t,\tilde{V}_m^{(n)}) - \frac{V(P,t-\tau)}{\tau} = -(c^* - f_u^{(n)})\,\tilde{Z}_m^{(n)}(P,t) \geqslant 0,$$

$$\tilde{V}_m^{(n)}(P,t) = \begin{cases} g(P,t), & P \in \gamma_m^{h0}; \\ V_m^{(n)}(P,t), & P \in \gamma_m^{hb}; \\ V_{m+1}^{(n)}(P,t), & P \in \gamma_m^{he}. \end{cases} \tag{20}$$

From (13), (14), $\bar{V}^{(n)}$ satisfies the boundary condition in (7). From (18), (20) and the definition of $\bar{V}^{(n)}$ in (15), we have

$$\mathscr{L}\bar{V}^{(n)}(P,t) + f(P,t,\bar{V}^{(n)}) - \frac{V(P,t-\tau)}{\tau} \geqslant 0,$$

$$P \in \Omega^h \setminus \gamma^h, \quad \gamma^h = \gamma^{hb} \cup \gamma^{he}, \quad \gamma^{hb,e} = \bigcup_{m=1}^{M-1} \gamma_m^{hb,e}.$$

Now, we prove that this inequality holds true on the interfacial boundaries $\gamma_m^{hb,e}$, $m = 1, \ldots, M-1$ and, hence, $\bar{V}^{(n)}$ is an upper solution to (7). We check this inequality in the case of the left boundary γ_m^{hb}, since the second case is checked in a similar way. From (13), (14) and (17), we conclude that the mesh function $W_m^{(n)} = V_m^{(n)} - \tilde{V}_m^{(n)}$ satisfies the difference problem

$$\mathscr{L}W_m^{(n)}(P,t) + c^* W_m^{(n)}(P,t) = 0, \qquad P \in \omega_m^{hb} = \Omega_m^h \cap \omega_m^h,$$
$$W_m^{(n)}(P,t) = 0, \quad P \in \partial\omega_m^{hb} \setminus \Gamma_m^{he}, \qquad W_m^{(n)}(P,t) \geqslant 0, \quad P \in \Gamma_m^{he}. \tag{21}$$

In view of the maximum principle in Lemma 1,

$$V_m^{(n)}(P,t) - \tilde{V}_m^{(n)}(P,t) \geqslant 0, \quad P \in \bar{\omega}_m^{hb}.$$

By (14), $V_m^{(n)}(P,t) = \tilde{V}_m^{(n)}(P,t)$, $P \in \gamma_m^{hb}$, and we get

$$-\mu^2 D_+^y D_-^y V_m^{(n)}(P,t) = -\mu^2 D_+^y D_-^y \bar{V}^{(n)}(P,t), \quad P \in \gamma_m^{hb},$$
$$-\mu^2 D_+^x D_-^x V_m^{(n)}(P,t) \leqslant -\mu^2 D_+^x D_-^x \bar{V}^{(n)}(P,t), \quad P \in \gamma_m^{hb}.$$

Thus, from here and (18), we conclude that

$$\mathscr{L}\bar{V}^{(n)}(P,t) + f(\bar{V}^{(n)}) - \frac{V(P,t-\tau)}{\tau} \geqslant \mathscr{L}V_m^{(n)}(P,t) + f(V_m^{(n)}) - \frac{V(P,t-\tau)}{\tau} \geqslant 0, \quad P \in \gamma_m^{hb}.$$

This leads to the fact that $\bar{V}^{(n)}$ is an upper solution of problem (7).

By (17) and (19), the sequence $\{\bar{V}^{(n)}\}$ is monotone decreasing and bounded by a lower solution. Indeed, if $\underline{V}$ is a lower solution, then by the definition of lower and upper solutions and the mean-value theorem, for $W^{(n)} = \bar{V}^{(n)} - \underline{V}$ we have

$$\mathscr{L}W^{(n)}(P,t) + f_u^{(n)}(P,t)W^{(n)}(P,t) \geqslant 0, \quad P \in \Omega^h,$$

$$W^{(n)}(P,t) \geqslant 0, \quad P \in \partial\Omega^h.$$

In view of the maximum principle in Lemma 1, it follows that $\underline{V} \leqslant \bar{V}^{(n)}$, $n \geqslant 0$. Thus, $\lim \bar{V}^{(n)} = \bar{V}$ as $n \to \infty$ exists and satisfies the relation

$$\bar{V}(P,t) \leqslant \bar{V}^{(n+1)}(P,t) \leqslant \bar{V}^{(n)}(P,t) \leqslant \bar{V}^{(0)}(P,t), \quad P \in \bar{\Omega}^h.$$

Now we prove the last point of this theorem that the limiting function $\bar{V}$ is the solution to (7), i.e., $\bar{V}(P,t)=\mathscr{V}(P,t),\ P\in\bar{\Omega}^h$. By (15)

$$\lim_{n\to\infty} Z_m^{(n)}(P,t)=\lim_{n\to\infty}[\bar{V}^{(n)}(P,t)-\bar{V}^{(n-1)}(P,t)]=0,$$

$$P\in\overline{\Omega_m^h\setminus(\omega_{m-1}^h\cup\omega_m^h)},$$

$$\lim_{n\to\infty}\tilde{Z}_m^{(n)}(P,t)=\lim_{n\to\infty}[\bar{V}^{(n)}(P,t)-\bar{V}^{(n-1)}(P,t)]=0,\quad P\in\bar{\omega}_m^h.$$

From here and letting $n\to\infty$ in (18) and (20), shows that $\bar{V}$ is the solution of (7) on $\Omega^h\setminus\gamma^h$. Now we verify that $\bar{V}$ satisfies (7) on the interfacial boundaries $\gamma_m^{hb,e}$, $m=1,\dots,M-1$. Since $V_m^{(n)}(P,t)-\tilde{V}_m^{(n)}(P,t)=\bar{V}^{(n-1)}(P,t)-\bar{V}^{(n)}(P,t),\ P\in\Gamma_m^{he}$, from (21) and (4) we conclude that

$$\lim_{n\to\infty}V_m^{(n)}(P,t)=\lim_{n\to\infty}\tilde{V}_m^{(n)}(P,t)=\bar{V}(P,t),\quad P\in\bar{\omega}_m^{hb}.$$

From here and (18), it follows that for $P\in\gamma_m^{hb}$

$$\lim_{n\to\infty}[\mathscr{L}\bar{V}^{(n)}(P,t)+f(\bar{V}^{(n)})]=\lim_{n\to\infty}[\mathscr{L}V_m^{(n)}(P,t)+f(V_m^{(n)})]=0,$$

and hence, $\bar{V}$ solves (7) on γ_m^{hb}. In a similar way, we can prove the last result on γ_m^{he}. This proves the theorem. □

Remark 6. The proposed algorithm (13)–(16) can be applied for solving "unperturbed" problems of form (1), i.e., in the case of $\mu=O(1)$. However, as we show below, this algorithm can be most efficiently used at small values of μ.

3.2. Convergence analysis of algorithm (13)–(16)

We now establish convergence properties of algorithm (13)–(16).

On mesh $\bar{\Omega}_*^h=\bar{\Omega}_*^{hx}\times\bar{\Omega}^{hy}$:

$$\bar{\Omega}_*^{hx}=\{x_i,i=0,1,\dots,N_x^*;x_0=x_a,x_{N_x^*}=x_b\},$$

where $x_a<x_b$, and $\bar{\Omega}^{hy}$ from (2), consider the following difference problems:

$$\mathscr{L}W(P)+c^*W(P)=F(P),\quad P\in\Omega_*^h,\ W(P)=W^0(P),\ P\in\partial\Omega_*^h, \tag{22}$$

and

$$\mathscr{L}\Phi^s(P)+c^*\Phi^s(P)=0,\quad P\in\Omega_*^h,$$

$$\Phi^s(P)=1,\quad P\in\Gamma^{hs},\ \Phi^s(P)=0,\ P\in\partial\Omega_*^h\setminus\Gamma^{hs},\ s=1,2,3,4, \tag{23}$$

where $\mathscr{L}$ from (3) and Γ^{hs} is the sth side of the rectangular mesh $\bar{\Omega}_*^h$. We suppose that

$$\Gamma^{h1}=\{x=x_a;\ y=y_j,\ 0\leqslant j\leqslant N_y\},\quad \Gamma^{h2}=\{x=x_b;\ y=y_j,\ 0\leqslant j\leqslant N_y\},$$

$$\Gamma^{h3}=\{x=x_i,\ 0\leqslant i\leqslant N_x^*;\ y=0\},\quad \Gamma^{h4}=\{x=x_i,\ 0\leqslant i\leqslant N_x^*;\ y=1\}.$$

Lemma 2. *If $W(P)$ and $\Phi^s(P)$, $s=1,2,3,4$ are the solutions to (22) and (23), respectively, then the following estimate holds true*:

$$|w(P)| \leqslant \sum_{s=1}^{4} \Phi^s(P)\|w^0\|_{\Gamma^{hs}} + \left[1-\sum_{s=1}^{4}\Phi^s(P)\right]\frac{\|F\|_{\Omega^h_*}}{c^*}, \quad P\in\bar{\Omega}^h_*. \tag{24}$$

The proof of the lemma can be found in [4].

Introduce the notation

$$\hbar^b_m = 2^{-1}(h^{b-}_m + h^{b+}_m), \quad \hbar^e_m = 2^{-1}(h^{e-}_m + h^{e+}_m),$$

where h^{b-}_m, h^{b+}_m are the mesh step sizes on the left and on the right from point x^b_m, respectively, and h^{e-}_m, h^{e+}_m are the mesh step sizes on the left and on the right from point x^e_m, respectively,

$$\kappa^b_m = \frac{\mu^2}{(c^*+\tau^{-1})\hbar^b_m h^{b+}_m}, \quad q^b_m = \|\Phi^{\mathrm{II}}_m\|_{\gamma^{hb+}_m},$$

$$\kappa^e_m = \frac{\mu^2}{(c^*+\tau^{-1})\hbar^e_m h^{e-}_m}, \quad q^e_m = \|\Phi^{\mathrm{I}}_m\|_{\gamma^{he-}_m},$$

$$\gamma^{hb\pm}_m = \{x = x^b_m \pm h^{b\pm}_m, 0\leqslant y\leqslant 1\}, \quad \gamma^{he\pm}_m = \{x = x^e_m \pm h^{e\pm}_m, 0\leqslant y\leqslant 1\},$$

where $\Phi^{\mathrm{II}}_m(P)$ is the solution to (23) on $\bar{\omega}^{hb}_m = \overline{\Omega^h_m \cap \omega^h_m}$ with $s=2$ and $\Phi^{\mathrm{I}}_m(P)$ is the solution to (23) on $\bar{\omega}^{he}_m = \overline{\Omega^h_{m+1} \cap \omega^h_m}$ with $s=1$.

Denote

$$Z^{(n)}(P,t) = V^{(n)}(P,t) - V^{(n-1)}(P,t), \quad P\in\bar{\Omega}^h.$$

From (15), it follows that on $\bar{\Omega}^h_m, m=1,\ldots,M, Z^{(n)}$ can be written down in the form

$$Z^{(n)}(P,t) = \begin{cases} \tilde{Z}^{(n)}_{m-1}(P,t), & x_{m-1}\leqslant x\leqslant x^e_{m-1}; \\ Z^{(n)}_m(P,t), & x^e_{m-1}\leqslant x\leqslant x^b_m; \\ \tilde{Z}^{(n)}_m(P,t), & x^b_m\leqslant x\leqslant x_m, \end{cases} \tag{25}$$

where for simplicity, we indicate the discrete domains only in the x-variable, i.e., $x_{m-1}\leqslant x\leqslant x^e_{m-1}$ means $\{x_{m-1}\leqslant x\leqslant x^e_{m-1}, 0\leqslant y\leqslant 1\}$, and assume that for $m=1,M$, the corresponding domains $x_0\leqslant x\leqslant x^e_0$ and $x^b_M\leqslant x\leqslant x_M$ are empty.

Theorem 4. *For algorithm (13)–(16), on each time level the following estimate holds true*:

$$\|Z^{(n)}(t_k)\|_{\bar{\Omega}^h} \leqslant (\rho+\lambda)\|Z^{(n-1)}(t_k)\|_{\bar{\Omega}^h}, \quad t_k\in\Omega^\tau,$$

$$\lambda = \max_{1\leqslant m\leqslant M-1}\{\kappa^b_m q^b_m; \kappa^e_m q^e_m\}, \tag{26}$$

where $Z^{(n)} = V^{(n)} - V^{(n-1)}$, $\rho = c^*/(c^*+\tau^{-1})$.

Proof. From (13) and (4), we conclude the estimate on $Z_m^{(n)}(P,t)$

$$\|Z_m^{(n)}(t)\|_{\bar{\Omega}_m^h} \leqslant \frac{1}{c^* + \tau^{-1}} \|G(t, V^{(n-1)})\|_{\Omega_m^h} \leqslant \frac{1}{c^* + \tau^{-1}} \|G(t, V^{(n-1)})\|_{\Omega^h},$$

$$G(P,t,V^{(n-1)}) \equiv \mathscr{L} V^{(n-1)}(P,t) + f(P,t,V^{(n-1)}) - \tau^{-1} V(P,t-\tau).$$

From here, (14) and (4), it follows that

$$\|\tilde{Z}_m^{(n)}(t)\|_{\bar{\omega}_m^h} \leqslant \max\left\{\|Z_m^{(n)}(t)\|_{\gamma_m^{hb}}; \|Z_{m+1}^{(n)}(t)\|_{\gamma_m^{he}}; \frac{1}{c^* + \tau^{-1}} \|G(t, V^{(n-1)})\|_{\omega_m^h}\right\}$$

$$\leqslant \frac{1}{c^* + \tau^{-1}} \|G(t, V^{(n-1)})\|_{\Omega^h}.$$

Thus, from here and (25), conclude

$$\|Z^{(n)}(t)\|_{\bar{\Omega}^h} \leqslant \frac{1}{c^* + \tau^{-1}} \|G(t, V^{(n-1)})\|_{\Omega^h}.$$

By (18) and (20),

$$G(P,t,V^{(n-1)}) = -[c^* - f_u^{(n-1)}(P,t)]Z^{(n-1)}(P,t),$$

$$P \in \Omega^h \setminus \gamma^h, \quad \gamma^h = \gamma^{hb} \cup \gamma^{he}, \quad \gamma^{hb,e} = \bigcup_{m=1}^{M-1} \gamma_m^{hb,e}$$

and we have

$$\|Z^{(n)}(t)\|_{\bar{\Omega}^h} \leqslant \max\left\{\rho\|Z^{(n-1)}(t)\|_{\bar{\Omega}^h}; \frac{1}{c^* + \tau^{-1}} \|G(t, V^{(n-1)})\|_{\gamma^h}\right\}. \tag{27}$$

Taking into account that for $m = 1, \ldots, M-1$,

$$V^{(n-1)}(P,t) = \begin{cases} V_{m+1}^{(n-1)}(P,t) = \tilde{V}_m^{(n-1)}(P,t), & P \in \gamma_m^{he}, \\ \tilde{V}_m^{(n-1)}(P,t), & P \in \gamma_m^{he-}, \\ V_{m+1}^{(n-1)}(P,t), & P \in \gamma_m^{he+}, \end{cases}$$

we have

$$G(P_m^e, t, V^{(n-1)}) = G(P_m^e, t, V_{m+1}^{(n-1)}) - \left(\frac{\mu^2}{\hbar_m^e h_m^{e-}}\right)[\tilde{V}_m^{(n-1)}(P_m^{e-}, t) - V_{m+1}^{(n-1)}(P_m^{e-}, t)],$$

$$P_m^e = (x_m^e, y_j) \in \gamma_m^{he}, \quad P_m^{e-} = (x_m^e - h_m^{e-}, y_j) \in \gamma_m^{he-}.$$

From here and (18), it follows that

$$\frac{1}{c^* + \tau^{-1}} \|G(t_k, V^{(n-1)})\|_{\gamma_m^{he}} \leqslant \rho\|Z^{(n-1)}(t_k)\|_{\bar{\Omega}^h}$$

$$+ \kappa_m^e \|V_{m+1}^{(n-1)}(t_k) - \tilde{V}_m^{(n-1)}(t_k)\|_{\gamma_m^{he-}}.$$

Estimating the solution of (21) by (24), on $\bar{\omega}_m^{he}$ we get

$$|V_{m+1}^{(n-1)}(P,t_k) - \tilde{V}_m^{(n-1)}(P,t_k)| \leqslant \Phi_m^{\mathrm{I}}(P)\|V_{m+1}^{(n-1)}(t_k) - \tilde{V}_m^{(n-1)}(t_k)\|_{\Gamma_m^{he}},$$

where $\Phi^{\mathrm{I}}_m(P)$ is the solution of (23) on $\bar{\omega}^{he}_m$. Since

$$V^{(n-1)}_{m+1}(P,t) - \tilde{V}^{(n-1)}_m(P,t) = V^{(n-2)}(P,t) - V^{(n-1)}(P,t), \quad P \in \Gamma^{he}_m$$

and $Z^{(n-1)} = V^{(n-1)} - V^{(n-2)}$, we conclude the estimate

$$\|V^{(n-1)}_{m+1}(t_k) - \tilde{V}^{(n-1)}_m(t_k)\|_{\gamma^{he-}_m} \leqslant q^e_m \|Z^{(n-1)}(t)\|_{\bar{\Omega}^h}. \tag{28}$$

Thus,

$$\frac{1}{c^* + \tau^{-1}} \|G(t_k, V^{(n-1)})\|_{\gamma^{he}_m} \leqslant (\rho + \kappa^e_m q^e_m)\|Z^{(n-1)}(t_k)\|_{\bar{\Omega}^h}.$$

Similarly, we can prove the estimate

$$\frac{1}{c^* + \tau^{-1}} \|G(t_k, V^{(n-1)})\|_{\gamma^{hb}_m} \leqslant (\rho + \kappa^b_m q^b_m)\|Z^{(n-1)}(t_k)\|_{\bar{\Omega}^h}.$$

Thus, from (27), we prove (26). □

Theorem 5. *Let $V^{(0)}(P,t)$ be an upper or lower solution in the domain decomposition algorithm (13)–(16), and let $f(P,t,u)$ satisfy (5). Suppose that on each time level, the number of iterates n_* satisfies $n_* \geqslant 2$. Then the following estimate on convergence rate holds*

$$\max_{1 \leqslant k \leqslant N_\tau} \|V(t_k) - U(t_k)\| \leqslant C(c^* + v)(\rho + \lambda)^{n_*-1},$$

$$v = (c^* + \tau^{-1})\lambda, \tag{29}$$

where ρ, λ are defined in Theorem 4, $U(P,t)$ is the solution to (2) and constant C is independent of τ. Furthermore, on each time level the sequence $\{V^{(n)}(P,t)\}$ converges monotonically.

Proof. Denote $W(P,t) = U(P,t) - V(P,t)$. From (2), (18) and (20) and taking into account (15) and (25), we get

$$\mathscr{L}W(P,t) + f_u(P,t)W(P,t) = [c^* - f^{(n_*)}_u(P,t)]Z^{(n_*)}(P,t) + \frac{W(P,t-\tau)}{\tau}, \quad P \in \Omega^h \setminus \gamma^h, \tag{30}$$

$$\mathscr{L}W(P,t) + f_u(P,t)W(P,t) = [c^* - f^{(n_*)}_u(P,t)]Z^{(n_*)}(P,t) + \Delta_m(P,t) + \frac{W(P,t-\tau)}{\tau}, \quad P \in \gamma^h, \tag{31}$$

$$W(P,t) = 0, \quad P \in \partial\Omega^h,$$

where

$$\Delta_m(x^b_m, y_j, t) = \frac{\mu^2}{\hbar^b_m h^{b+}_m} [\tilde{V}^{(n_*)}_m(P^{b+}_m, t) - V^{(n_*)}_m(P^{b+}_m, t)],$$

$$\Delta_m(x^e_m, y_j, t) = \frac{\mu^2}{\hbar^e_m h^{e-}_m} [\tilde{V}^{(n_*)}_m(P^{e-}_m, t) - V^{(n_*)}_{m+1}(P^{e-}_m, t)],$$

$$P^{b+}_m = (x^b_m + h^b_m, y_j) \in \gamma^{hb+}_m, \quad P^{e-}_m = (x^e_m - h^{e-}_m, y_j) \in \gamma^{he-}_m.$$

Using (28) and the similar estimate on γ_m^{hb+}

$$\|V_m^{(n)}(t_k) - \tilde{V}_m^{(n)}(t_k)\|_{\gamma_m^{hb+}} \leqslant q_m^b \|Z^{(n)}(t_k)\|_{\bar{\Omega}^h},$$

we get the estimate

$$\max_{1 \leqslant m \leqslant M-1} \{\|\Delta_m(t_k)\|_{\gamma^h}\} \leqslant v\|Z^{(n_*)}(t_k)\|_{\bar{\Omega}^h}.$$

From here, (30), (31) and using (4), we obtain the estimate

$$\|W(t_k)\|_{\bar{\Omega}^h} \leqslant \tau(c^* + v)\|Z^{(n_*)}(t_k)\|_{\bar{\Omega}^h} + \|W(t_k - \tau)\|_{\bar{\Omega}^h}.$$

Using (26), we prove by induction the estimates

$$\|W(t_k)\| \leqslant \left(\sum_{l=1}^{k} C_l\right) \tau(c^* + v)(\rho + \lambda)^{n_*-1}, \quad k = 1, \dots, N_\tau,$$

$$\|Z^{(1)}(t_l)\|_{\bar{\Omega}^h} \leqslant \tau\|\mathscr{L}V^{(0)}(t_l) + f(V^{(0)}) - \tau^{-1}V(t_l - \tau)\| \leqslant C_l,$$

where all constants C_l are independent of τ. Since $N_\tau \tau = T$, we prove the estimate in the theorem with $C = TC_0$, where $C_0 = \max_{1 \leqslant l \leqslant N_\tau} C_l$. □

3.3. Estimates on the rate of convergence of algorithm (13)–(16)

Here we analyse a convergence rate of algorithm (13)–(16) applied to the difference scheme (2) defined on meshes of the general type introduced in [17]. On these meshes, the difference scheme (2) converges μ-uniformly to the solution of (1).

A mesh of this type is formed in the following manner. We divide each of the intervals $\bar{\Omega}^x = [0, 1]$ and $\bar{\Omega}^y = [0, 1]$ into three parts $[0, \sigma_x]$, $[\sigma_x, 1 - \sigma_x]$, $[1 - \sigma_x, 1]$ and $[0, \sigma_y]$, $[\sigma_y, 1 - \sigma_y]$, $[1 - \sigma_y, 1]$, respectively. Assuming that N_x, N_y are divisible by 4, in the parts $[0, \sigma_x]$, $[1 - \sigma_x, 1]$ and $[0, \sigma_y]$, $[1 - \sigma_y, 1]$ we allocate $N_x/4 + 1$ and $N_y/4 + 1$ mesh points, respectively, and in the parts $[\sigma_x, 1 - \sigma_x]$ and $[\sigma_y, 1 - \sigma_y]$ we allocate $N_x/2 + 1$ and $N_y/2 + 1$ mesh points, respectively. Points σ_x, $(1 - \sigma_x)$ and σ_y, $(1 - \sigma_y)$ correspond to transition to the boundary layers. We consider meshes $\bar{\Omega}^{hx}$ and $\bar{\Omega}^{hy}$ which are equidistant in $[x_{N_x/4}, x_{3N_x/4}]$ and $[y_{N_y/4}, y_{3N_y/4}]$ but graded in $[0, x_{N_x/4}]$, $[x_{3N_x/4}, 1]$ and $[0, y_{N_y/4}]$, $[y_{3N_y/4}, 1]$. On $[0, x_{N_x/4}]$, $[x_{3N_x/4}, 1]$ and $[0, y_{N_y/4}]$, $[y_{3N_y/4}, 1]$ let our mesh be given by a mesh generating function ϕ with $\phi(0) = 0$ and $\phi(1/4) = 1$ which is supposed to be continuous, monotonically increasing, and piecewise continuously differentiable. Then our mesh is defined by

$$x_i = \begin{cases} \sigma_x \phi(t_i), & t_i = i/N_x, \ i = 0, \dots, N_x/4; \\ ih_x, & i = N_x/4 + 1, \dots, 3N_x/4 - 1; \\ 1 - \sigma_x(1 - \phi(t_i)), & t_i = (i - 3N_x/4)/N_x, \ i = 3N_x/4 + 1, \dots, N_x, \end{cases}$$

$$y_j = \begin{cases} \sigma_y \phi(t_j), & t_j = j/N_y, \ j = 0, \dots, N_y/4; \\ jh_y, & j = N_y/4 + 1, \dots, 3N_y/4 - 1; \\ 1 - \sigma_y(1 - \phi(t_j)), & t_j = (j - 3N_y/4)/N_y, \ j = 3N_y/4 + 1, \dots, N_y, \end{cases}$$

$$h_x = 2(1 - 2\sigma_x)N_x^{-1}, \quad h_y = 2(1 - 2\sigma_y)N_y^{-1}.$$

We also assume that ϕ' does not decrease. This condition implies that

$$h_{xi} \leqslant h_{x,i+1},\ i = 1,\dots,N_x/4-1, \quad h_{xi} \geqslant h_{x,i+1},\ i = 3N_x/4+1,\dots,N_x-1,$$

$$h_{yj} \leqslant h_{y,j+1},\ j = 1,\dots,N_y/4-1, \quad h_{yj} \geqslant h_{y,j+1},\ j = 3N_y/4+1,\dots,N_y-1.$$

3.3.1. Shishkin-type mesh

We choose the transition points σ_x, $(1-\sigma_x)$ and σ_y, $(1-\sigma_y)$ in Shishkin's sense (see [13] for details), i.e.,

$$\sigma_x = \min\{4^{-1}, v_1\mu \ln N_x\}, \quad \sigma_y = \min\{4^{-1}, v_2\mu \ln N_y\},$$

where v_1 and v_2 are positive constants. If $\sigma_{x,y} = 1/4$, then $N_{x,y}^{-1}$ are very small relative to μ. This is unlikely in practice, and in this case the difference scheme (2) can be analysed using standard techniques. We therefore assume that

$$\sigma_x = v_1\mu \ln N_x, \quad \sigma_y = v_2\mu \ln N_y.$$

Consider the mesh generating function ϕ in the form

$$\phi(t) = 4t.$$

In this case the meshes $\bar{\Omega}^{hx}$ and $\bar{\Omega}^{hy}$ are piecewise equidistant with the step sizes

$$N_x^{-1} < h_x < 2N_x^{-1}, \quad h_{x\mu} = 4v_1\mu N_x^{-1}\ln N_x,$$

$$N_y^{-1} < h_y < 2N_y^{-1}, \quad h_{y\mu} = 4v_2\mu N_y^{-1}\ln N_y. \tag{32}$$

The difference scheme (2) on the piecewise uniform mesh (32) converges μ-uniformly to the solution of (1):

$$\max_{(P,t)\in\bar{\Omega}^h\times\bar{\Omega}^\tau} |U(P,t)-u(P,t)| \leqslant C((N^{-1}\ln N)^2+\tau), \quad N = \min\{N_x, N_y\},$$

where constant C is independent of ε, N and τ. The proof of this result can be found in [13].

To estimate the rate of convergence in Theorem 5, we have to estimate λ in (26). Introduce the one-dimensional difference problems

$$-\mu^2 D_+^x D_-^x \varphi_m^{\mathrm{I}} + \tau^{-1}\varphi_m^{\mathrm{I}} = 0,\ x_m < x_i < x_m^e, \quad \varphi_m^{\mathrm{I}}(x_m) = 1,\ \varphi_m^{\mathrm{I}}(x_m^e) = 0,$$

$$-\mu^2 D_+^x D_-^x \varphi_m^{\mathrm{II}} + \tau^{-1}\varphi_m^{\mathrm{II}} = 0,\ x_m^b < x_i < x_m, \quad \varphi_m^{\mathrm{II}}(x_m^b) = 0,\ \varphi_m^{\mathrm{II}}(x_m) = 1.$$

The solutions of these problems on the uniform mesh with the step size h can be written in the forms

$$\varphi_m^{\mathrm{I}}(x_i) = \frac{r_1^{N_m^1} r_2^i - r_2^{N_m^1} r_1^i}{r_1^{N_m^1} - r_2^{N_m^1}} \leqslant r_1^{-i},\ i = 0,\dots,N_m^1, \quad x_0 = x_m,\ x_{N_m^1} = x_m^e,$$

$$\varphi_m^{\mathrm{II}}(x_i) = \frac{r_1^i - r_2^i}{r_1^{N_m^2} - r_2^{N_m^2}} \leqslant r_1^{i-N_m^2},\ i = 0,\dots,N_m^2, \quad x_0 = x_m^b,\ x_{N_m^2} = x_m,$$

$$r_{1,2} = (1+p) \pm [(1+p)^2-1]^{1/2}, \quad p = \frac{h^2}{2\tau\mu^2},$$

where N_m^1+1 and N_m^2+1 are the numbers of mesh points on the intervals $[x_m, x_m^e]$ and $[x_m^b, x_m]$, respectively.

Lemma 3. *The following estimates hold true*:

$$\Phi_m^{\mathrm{I}}(P) \leqslant \varphi_m^{\mathrm{I}}(x), \quad P=(x,y)\in\bar{\omega}_m^{he}=\overline{\Omega_{m+1}^h\cap\omega_m^h},$$

$$\Phi_m^{\mathrm{II}}(P) \leqslant \varphi_m^{\mathrm{II}}(x), \quad P=(x,y)\in\bar{\omega}_m^{hb}=\overline{\Omega_m^h\cap\omega_m^h},$$

where $\Phi_m^{\mathrm{I}}(P)$ is the solution to (23) on $\bar{\omega}_m^{he}$ with $s=1$ and $\Phi_m^{\mathrm{II}}(P)$ is the solution to (23) on $\bar{\omega}_m^{hb}$ with $s=2$.

Proof. We prove the second inequality, since the first one is checked in a similar manner. From the maximum principle, it follows that $\Phi_m^{\mathrm{II}}(P)\geqslant 0$, $\varphi_m^{\mathrm{II}}(x)\geqslant 0$. From (23), we conclude that the difference $\psi_m(P)=\varphi_m^{\mathrm{II}}(x)-\Phi_{m,k}^{\mathrm{II}}(P)$ satisfies the difference problem

$$\mathscr{L}\psi_m(P)=c^*\Phi_m^{\mathrm{II}}(P), \quad P\in\omega_m^{hb},\ \psi_m(P)\geqslant 0,\ P\in\partial\omega_m^{hb}.$$

Since the right-hand side of the difference equation is nonnegative, by the maximum principle, we conclude $\psi_m\geqslant 0$. □

Thus, on the uniform mesh we have the estimate

$$\Phi_m^{\mathrm{I}}(x_i,y_j)\leqslant\varphi_m^{\mathrm{I}}(x_i)\leqslant r_1^{-i}, \quad (x_i,y_j)\in\bar{\omega}_m^{he},$$

$$\Phi_m^{\mathrm{II}}(x_i,y_j)\leqslant\varphi_m^{\mathrm{II}}(x_i)\leqslant r_1^{i-N_m^2}, \quad (x_i,y_j)\in\bar{\omega}_m^{hb}$$

and we can estimate q_m^b and q_m^e in (26) by

$$q_m^b=\|\Phi_m^{\mathrm{II}}\|_{\gamma_m^{hb+}}\leqslant\varphi_m^{\mathrm{II}}(x_1)\leqslant r_1^{1-N_m^2},$$

$$q_m^e=\|\Phi_m^{\mathrm{I}}\|_{\gamma_m^{he-}}\leqslant\varphi_m^{\mathrm{I}}(x_{N_m^1-1})\leqslant r_1^{1-N_m^1}. \tag{33}$$

Consider algorithm (13)–(16) on the piecewise uniform mesh (32) with the interfacial subdomains $\bar{\omega}_m^h$, $m=1,\ldots,M-1$ located in the x-direction outside the boundary layer, where the step size h_x from (32) is in use. Assume that $N_m^{1,2}\geqslant 2$, from (33) we have

$$\max_{1\leqslant m\leqslant M-1}\{q_m^{b,e}\}\leqslant\frac{1}{r_1}. \tag{34}$$

From here and taking into account that $\kappa_m^{b,e}\leqslant\tau\mu^2/h_x^2$, we estimate λ in (26) by $\lambda\leqslant 1/(2r_1p)$. If $\mu\leqslant h_x$, then $1/r_1\leqslant 1/(2p)$ and we conclude that

$$\lambda\leqslant\left(\frac{\tau^{1/2}\mu}{h_x}\right)^4\leqslant\tau^2.$$

Thus, the right-hand side in (29) is estimated by

$$C(c^*+v)(\rho+\lambda)^{n_*-1}\leqslant\tilde{C}(\rho+\tau^2)^{n_*-1},$$

where constant $\tilde{C}$ is independent of τ.

Remark 7. We mention that the implicit difference scheme (2) is of the first order with respect to τ and $\rho=c^*/(c^*+\tau^{-1})\leqslant c^*\tau$. Thus, to guarantee the consistency of the global errors in the difference scheme (2) and in the monotone domain decomposition algorithm (13)–(16), one would enough choose $n_*=2$.

3.3.2. Bakhvalov-type mesh

We choose the transition points σ_x, $(1-\sigma_x)$ and σ_y, $(1-\sigma_y)$ in Bakhvalov's sense (see [2] for details), i.e.,

$$\sigma_x = v_1 \mu \ln(1/\mu), \quad \sigma_y = v_2 \mu \ln(1/\mu),$$

and the mesh generating function ϕ is given in the form

$$\phi(t) = \frac{\ln[1 - 4(1-\mu)t]}{\ln \mu}.$$

The difference scheme (2) on the Bakhvalov-type mesh converges μ-uniformly to the solution of (1):

$$\max_{(P,t)\in\bar{\Omega}^h\times\bar{\Omega}^\tau} |U(P,t) - u(P,t)| \leqslant C(N^{-1} + \tau), \quad N = \min\{N_x, N_y\},$$

where constant C is independent of μ, N and τ. The proof of this result can be found in [3].

If the interfacial subdomains $\bar{\omega}_m^h$, $m = 1, \ldots, M-1$ are located in the x-direction outside the boundary layers, then for algorithm (13)–(16) on the Bakhvalov-type mesh, the estimates on λ and on the right-hand side in (29) are the same as for the Shishkin-type mesh, and Remark 7 holds true.

3.3.3. Modified piecewise equidistant mesh

Now we modify the piecewise equidistant mesh of Shishkin-type in the x-direction. Let the number of mesh points $N_{x\mu}$ and the step size $h_{x\mu}$ in the boundary layers be chosen in the form

$$N_{x\mu} = \alpha \ln(1/\mu), \quad h_{x\mu} = v\mu, \tag{35}$$

where α and v are positive constants. In this case, the transition points σ_x and $(1-\sigma_x)$ are defined by

$$\sigma_x = h_{x\mu} N_{x\mu} = (\alpha v)\mu \ln(\mu^{-1}).$$

We note that, in general, the difference scheme (2) on the modified piecewise equidistant mesh (32), (35) does not converge μ-uniformly to the solution of (1).

Consider algorithm (13)–(16) on the modified mesh (32), (35) with the first and last M_μ interfacial subdomains $\bar{\omega}_m^h$, $m = 1, \ldots, M_\mu$, $m = M - M_\mu, \ldots, M-1$ located in the x-direction inside the boundary layers, where the step size $h_{x\mu}$ from (35) is in use. Taking into account $q_m^{b,e} \leqslant 1$ in (26) and assuming $\mu \leqslant h_x$, we estimate λ in (26) by

$$\lambda \leqslant \max\left\{ \frac{\tau\mu^2}{(h_{x\mu})^2}; \frac{\tau\mu^2}{(h_x)^2} \right\} \leqslant \tau \max\{v^{-2}; 1\}.$$

Thus, the right-hand side in (29) is estimated by

$$C(c^* + v)(\rho + \lambda)^{n_*-1} \leqslant \tilde{C}(\rho + \tau)^{n_*-1},$$

where constant $\tilde{C}$ is independent of τ.

We mention that Remark 7 holds true for the monotone domain decomposition algorithm (13)–(16) on the modified mesh (32), (35).

4. Numerical experiments

Consider problem (1) with $f(P,t,u)=(u-4)/(5-u)$, $g(P,t)=1$, which models the biological Michaelis–Menton process without inhibition [6]. This problem gives

$$c^*=1, \quad \bar{V}^{(0)}(P,\tau)=\begin{cases}4, & P\in\Omega^h,\\ 1, & P\in\partial\Omega^h,\end{cases} \qquad \underline{V}^{(0)}(P,\tau)=\begin{cases}0, & P\in\Omega^h,\\ 1, & P\in\partial\Omega^h,\end{cases}$$

where $\bar{V}(P,\tau)$ and $\underline{V}(P,\tau)$ are the upper and lower solutions on the time level $t_1=\tau$ corresponding to $u^0(P)=\bar{V}^{(0)}(P,\tau)$, $P\in\bar{\Omega}^h$.

Consider the case of the upper sequence in algorithm (13)–(16). For this test problem, we may use the solution $\bar{V}(P,t-\tau)$ as an initial guess $\bar{V}^{(0)}(P,t)$ in the monotone domain decomposition algorithm (13)–(16). Indeed, the following lemma holds true.

Lemma 4. *If the functions f and g in* (1) *are independent of t, then the solution $\bar{V}(P,t-\tau)$ on time level $t-\tau$ of the monotone domain decomposition algorithm* (13)–(16) *is the upper solution on the next time level t, i.e.,*

$$\mathscr{L}\bar{V}(P,t-\tau)+f(P,\bar{V}(P,t-\tau))-\frac{\bar{V}(P,t-\tau)}{\tau}\geqslant 0, \quad P\in\Omega^h.$$

Proof. On time level $t_2=2\tau$, for $\bar{V}^{(0)}(P,2\tau)=\bar{V}(P,\tau)$, we have

$$\mathscr{L}\bar{V}(P,\tau)+f(\bar{V}(P,\tau))-\frac{\bar{V}(P,\tau)}{\tau}\geqslant \mathscr{L}\bar{V}(P,\tau)+f(\bar{V}(P,\tau))-\frac{u^0(P)}{\tau}\geqslant 0,$$

where we have taken into account that the sequence $\{\bar{V}^{(n)}(P,\tau)\}$ is monotone decreasing, and, hence,

$$\bar{V}(P,\tau)=\bar{V}^{(n_*)}(P,\tau)\leqslant u^{(0)}(P), \quad P\in\bar{\Omega}^h.$$

Since $\bar{V}(P,\tau)$ satisfies the boundary condition, we conclude that $\bar{V}(P,\tau)$ is the upper solution on the time level t_2, $k=2$. Now by induction on k, we prove the required result. □

On each time level t_k, the stopping criterion is chosen in the form

$$\|V^{(n)}(t_k)-V^{(n-1)}(t_k)\|_{\bar{\Omega}^h}\leqslant\delta,$$

where $\delta=10^{-5}$. All the discrete linear systems are solved by *ICCG*-method.

It is found that in all the numerical experiments the basic feature of monotone convergence of the upper and lower sequences is observed. In fact, the monotone property of the sequences holds at every mesh point in the domain. This is, of course, to be expected from the analytical consideration.

Consider the monotone domain decomposition algorithm (13)–(16) on the uniform mesh with $N_x=N_y$. The interfacial subdomains $\bar{\omega}_m^h$, $m=1,\dots,M-1$ contain only three mesh points in the x-direction. In Table 1, for $\tau_1=5\times10^{-2}$, $\tau_2=10^{-2}$, $M=32$ and $\mu=10^{-2}$, 10^{-3}, and for various values of N_x, we give the average (over ten time levels) numbers of iterations $n_{\tau1}$, $n_{\tau2}$ required to satisfy the stopping criterion. From the data, it follows that for $\mu\leqslant h_x=1/N_x$ the numbers of iterations are equal to the numbers of iterations for the undecomposed monotone algorithm with $M=1$. These numerical results confirm our theoretical estimates.

Table 1
Average numbers of iterations n_{τ_1}, n_{τ_2} on the uniform mesh

μ	n_{τ_1}; n_{τ_2}			
10^{-2}	3.6; 2.6	4.0; 3.0	4.7; 3.0	6.6; 3.0
10^{-3}	2.0; 2.0	2.0; 2.0	2.9; 2.0	3.0; 2.0
N_x	64	128	256	512

Table 2
Average numbers of iterations n_{τ_1}, n_{τ_2} for $\mu = 10^{-1}$ on the uniform mesh

M	n_{τ_1}; n_{τ_2}		
4	5.9; 3.0	10.9; 3.0	18.8; 3.0
8	10.6; 5.0	19.3; 5.7	36.4; 9.2
16	12.6; 5.8	23.1; 9.6	43.5; 17.2
32	18.9; 7.0	32.9; 11.0	60.1; 20.5
N_x	64	128	256

Table 3
Average numbers of iterations n_{τ_1}, n_{τ_2} on the piecewise uniform mesh (32)

μ	n_{τ_1}; n_{τ_2}		
10^{-2}	3.8; 2.8	4.0; 3.0	6.6; 3.0
10^{-3}	3.8; 2.8	4.0; 3.0	4.0; 3.0
N_x	64	128; 256	512

Table 2 shows the numerical experiments for $\mu = 10^{-1}$ and for various values of N_x, where we violate the condition $\mu \leqslant h_x$. In this case, the average numbers of iterations are monotone increasing functions of M and N_x.

Now, consider the monotone domain decomposition algorithm (13)–(16) on the piecewise uniform mesh (32) with $N_x = N_y$. The interfacial sub-domains $\bar{\omega}_m^h$, $m = 1, \ldots, M-1$ contain only three mesh points in the x-direction. In Table 3, for $\mu = 10^{-2}$, 10^{-3} and for various values of N_x, we give the average (over 10 time levels) numbers of iterations n_{τ_1}, n_{τ_2} ($\tau_1 = 5 \times 10^{-2}, \tau_2 = 10^{-2}$) required to satisfy the stopping criterion. The μ-dependence of the step sizes h_x and $h_{x\mu}$ of the piecewise uniform mesh (32) is tabulated in Table 4. Since for our data set we allow $\sigma_x > 0.25$, the step size $h_{x\mu}$ is calculated as

$$h_{x\mu} = \frac{4 \min\{0.25, \sigma_x\}}{N_x}.$$

Table 4
The μ-dependence of h_x, $h_{x\mu}$

μ	$h_x; h_{x\mu}$		
10^{-2}	1.82E−02; 1.30E−02	8.03E−03; 7.59E−03	1.95E−03; 1.95E−03
10^{-3}	2.99E−02; 1.30E−03	1.49E−02; 7.59E−04	2.44E−04; 3.66E−03
N_x	64	128	512

We mention that n_{τ_1}, n_{τ_2} are independent of the number of subdomains M. From the data presented in Tables 3 and 4, it follows that if the condition $\mu \lessapprox h_x$ holds true then the numbers of iterations are equal to the numbers of iterations for the undecomposed monotone algorithm with $M=1$. If we violate this condition as in the case with $\mu=10^{-2}$, $N_x=512$ and $\tau_1=5\times10^{-2}$, then the number of iterations n_{τ_1} exceeds the number of iteration for the undecomposed monotone algorithm. Thus, the numerical experiments confirm our theoretical estimates that the monotone domain decomposition algorithm (13)–(16) can be most efficiently used if the condition $\mu \lessapprox h_x$ holds true.

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